

Three pearls of Bernoulli numbers

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Abstract

The Bernoulli numbers are fascinating and ubiquitous numbers; they occur in several domains of Mathematics like Number theory (FLT), Group theory, Calculus and even in Physics. Since Bernoulli's work, they are yet studied to understand their deep nature [9], [6] and particularly to find relationships between them. In this paper, we give, firstly, a short response [15] to a problem stated, in 1971, by Carlitz [4] and studied by many authors like Prodinger [10]; the second pearl is an answer to a question raised, in 2008, by Tom Apostol [1]. The third pearl is another proof of a relationship already given in 2011, by the authors [14]

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1 Introduction

The aim of this work is to give original proofs of three relationships involving Bernoulli numbers. In the first section, we give a short proof to a problem stated, in 1971, by Carlitz [4] and studied by many authors like Prodinger [10]. In the second section, we give a response to a question raised by Apostol in 2008 in his relevant paper [1]. In the third section, we expose a different proof of a relationship already given by us in 2011 [15].

2 Pearl #1: Carlitz's Problem

In Mathematics Magazine, Vol. 44, No. 2 (Mar., 1971), pp. 105-114+101, Carlitz states the following problem :

define $\{B_n\}$ by means of $B_0 = 1$ and for $n > 1$

$$\sum_{k=0}^n \binom{n}{k} B_k = B_n$$

show that for arbitrary $m, n > 0$

$$(-1)^m \sum_{k=0}^m \binom{m}{k} B_{n+k} = (-1)^n \sum_{k=0}^n \binom{n}{k} B_{m+k}$$

This identity was firstly proved by Shanon [11] in 1971, by Gessel [7] in 2003, by Wu, Sun and Pan [13] in 2004, by Vassilev-Missana [12] in 2005, by Chen and Sun [5] in 2009, by Gould and J. Quaintance [8] in 2014 and by Prodinger [10] in 2014. The Prodinger's proof is very short and uses a two variables formal series. In fact, one can see that Carlitz's problem can be easily deduced from the following relationship already proved in 2012 by Bencherif and Garici in 2012 [3] :

$$(-1)^m \sum_{k=0}^{m+q} \binom{m+q}{k} \binom{n+q+k}{q} B_{n+k} - (-1)^{n+q} \sum_{k=0}^{n+q} \binom{n+q}{k} \binom{m+q+k}{q} B_{m+k} = 0.$$

Hereafter, we give a proof different from that was given by Prodinger.

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Proof. We consider the linear functional L defined on $\mathbb{Q}[x]$ by $L(x^n) = B_n$ for $n \geq 0$, which gives

$$L\left(\left(x + \frac{1}{2}\right)^{2n+1}\right) = B_{2n+1}\left(\frac{1}{2}\right) = 0,$$

see [1], p.182, then the polynomial defined by :

$$P(x) = (-1)^{m+q}x^{n+q}(1+x)^{m+q} - (-1)^n x^{m+q}(1+x)^{n+q}$$

satisfies

$$P\left(-\frac{1}{2} + x\right) + (-1)^q P\left(-\frac{1}{2} - x\right) = 0$$

and

$$P^{(q)}\left(-\frac{1}{2} + x\right) + P^{(q)}\left(-\frac{1}{2} - x\right) = 0$$

Now, with use of the equality :

$$L\left(\left(x + \frac{1}{2}\right)^{2n+1}\right) = B_{2n+1}\left(\frac{1}{2}\right) = 0.$$

And as $P^{(q)}$ is an even polynomial , we get :

$$L\left(P^{(q)}(x)\right) = 0$$

Thus

$$\frac{1}{q!}P^{(q)}(x) = (-1)^m \sum_{k=0}^{m+q} \binom{m+q}{k} \binom{n+q+k}{q} x^{n+k} - (-1)^{n+q} \sum_{k=0}^{n+q} \binom{n+q}{k} \binom{m+q+k}{q} x^{m+k} = 0$$

and finally:

$$(-1)^m \sum_{k=0}^{m+q} \binom{m+q}{k} \binom{n+q+k}{q} B_{n+k} - (-1)^{n+q} \sum_{k=0}^{n+q} \binom{n+q}{k} \binom{m+q+k}{q} B_{m+k} = 0$$

which yields the identity wanted by Carlitz, by taking $q = 0$. □

3 Pearl #2: APOSTOL'S PROBLEM

In his relevant paper published in 2008, Tom Apostol writes : *we leave it as a challenge to the reader to find another proof of (42) as a direct consequence of (3) without the use of integration.* In Apostol's paper, (42) denotes the relationship:

$$\sum_{k=0}^n \binom{n}{k} \frac{B_k}{(n+2-k)} = \frac{B_{n+1}}{n+1}, \quad n \geq 1$$

and (3) is one of the *six* definitions of the Bernoulli numbers he recalls to show his relationship and which is :

$$B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad \text{for } n \geq 2$$

As he said it, Apostol uses *integration method* to deduce his (42)- numbered relation from the Bernoulli numbers's definition that he has chosen. To take up the challenge he has launched, we expose a proof without use of *integration method*.

Proof. (Answer to Apostol's problem)

Let's define the sequence (u_n) by:

$$u_n := \sum_{k=0}^n \binom{n+1}{k} B_k$$

We can see that $u_n = 0$ for $n \geq 1$. Writing:

$$\binom{n}{k} \frac{1}{n+2-k} = \frac{1}{n+1} \binom{n+1}{k} - \frac{1}{(n+1)(n+2)} \binom{n+2}{k}$$

we get :

$$\sum_{k=0}^n \binom{n}{k} \frac{B_k}{n+2-k} = \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k} B_k - \frac{1}{(n+1)(n+2)} \sum_{k=0}^n \binom{n+2}{k} B_k$$

which yields :

$$\sum_{k=0}^n \binom{n}{k} \frac{B_k}{n+2-k} = \frac{1}{n+1} u_n - \frac{1}{(n+1)(n+2)} \left(u_{n+1} - \binom{n+2}{n+1} B_{n+1} \right)$$

As if $n \geq 1$, we have $u_n = u_{n+1} = 0$ and as $\binom{n+2}{n+1} = n+2$, we get :

$$\frac{-1}{(n+1)(n+2)} \left(-\binom{n+2}{n+1} B_{n+1} \right) = \frac{B_{n+1}}{n+1}.$$

which gives the asked relation :

$$\sum_{k=0}^n \binom{n}{k} \frac{B_k}{(n+2-k)} = \frac{B_{n+1}}{n+1}, \quad n \geq 1$$

Finally, Apostol's relationship is proved without use of *integration method*. □

4 Pearl #3: New proof of a relationship

In our paper [15], we proved the following relationship:

$$\sum_{k=0}^{n+q} \binom{n+q}{k} \left(\prod_{j=1}^q (n+k+j) \right) B_{n+k} = 0$$

where q is an odd number. For this, we showed that the two well-suited polynomials :

$$H_n(x) = \frac{1}{2} x^{n+q} (x-1)^{n+q}$$

and

$$K_n(x) = \sum_{k=0}^{n+q} \frac{\varepsilon_{n+k}}{n+q+k+1} \binom{n+q}{k} B_{n+q+k+1}(x) - B_{n+q+k+1}$$

Are equal, where $B_n(x)$ and B_n are respectively the Bernoulli polynomials and the Bernoulli numbers defined by the generating function:

$$\frac{x}{e^x - 1} e^{xz} = \sum_{n=0}^{+\infty} B_n(x) \frac{x^n}{n!}$$

knowing that $B_n = B_n(0) = B_n(1)$, $n \geq 2$; $B_{2n+1} = 0$, $n \geq 1$, see [1] , relations (11), (12) (13) and (15). Furthermore, we shall use the well-known equalities:

$$B_n(x+1) - B_n(x) = nx^{n-1}, \quad B'_n(x) = nB_{n-1}(x), \quad \int_x^{x+1} B_n(t)dt = x^n, \quad n \geq 0$$

for $n \geq 1$, see e.g. [1], relations (14), (27) and (30).

Now, to give another proof of the relation already proved in [15], we consider the polynomials:

$$\begin{aligned} P_n(x) &:= \frac{1}{2}x^{n+1}(x-1)^{n+1} \\ K_n(x) &:= \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1 - (-1)^{n+1-k}}{2} B_{n+1+k}(x) \\ H_n(x) &:= \frac{1}{2}(n+1)x^n(x-1)^n(2x-1) \end{aligned}$$

and the automorphism of the \mathbb{Q} -space vector $\mathbb{Q}[x]$ defined by $f(P(x)) = \int_x^{x+1} P(t)dt$.

First of all, let's prove the

Theorem 4.1. *The two polynomials $K_n(x)$ and $H_n(x)$ are equal, i.e. $K_n(x) = H_n(x)$.*

Proof.

$$\begin{aligned} \int_x^{x+1} P'_n(t)dt &= P_n(x+1) - P_n(x) \\ &= \frac{1}{2}x^{n+1}(x+1)^{n+1} - \frac{1}{2}x^{n+1}(x-1)^{n+1} \\ &= \frac{1}{2}x^{n+1}((x+1)^{n+1} - (x-1)^{n+1}) \\ &= x^{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{(1 - (-1)^{n+1-k})}{2} x^k \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1 - (-1)^{n+1-k}}{2} x^{n+1+k} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1 - (-1)^{n+1-k}}{2} \int_x^{x+1} B_{n+1+k}(t)dt \\ &= \int_x^{x+1} \left(\sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1 - (-1)^{n+1-k}}{2} B_{n+1+k}(t) \right) dt \end{aligned}$$

Thus, we can see that

$$f(P'_n(t)) = f\left(\sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1 - (-1)^{n+1-k}}{2} B_{n+1+k}(t)\right)$$

As f is bijective, we get :

$$P'_n(t) = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1 - (-1)^{n+1-k}}{2} B_{n+1+k}(t)$$

i.e.

$$P'_n(t) = K_n(t)$$

Let's compute $P'_n(x)$:

$$\begin{aligned} P'_n(x) &= \frac{1}{2}(x^2 - x)^{n+1}' \\ &= \frac{1}{2}(n+1)(x^2 - x)^n(2x - 1) \\ &= \frac{1}{2}(n+1)x^n(x-1)^n(2x - 1) \\ &= H_n(x) \end{aligned}$$

i.e.

$$K_n(x) = H_n(x).$$

□

Theorem 4.2. *The following identity holds:*

$$\sum_{k=0}^{n+q} \binom{n+q}{k} \left(\prod_{j=1}^q (n+k+j) \right) B_{n+k} = 0$$

Proof. To get this, let's replace n by $n+q-1$, $q \geq 1$, q odd. Then we compute the coefficient of x^q in the equality: $K_{n+q-1}(x) = H_{n+q-1}(x)$. The coefficient of x^q in the polynomial $K_n(x)$

$$K_n(x) := \sum_{k=0}^{n+q} \binom{n+q}{k} \frac{1 - (-1)^{n+q-k}}{2} B_{n+1+k}(x)$$

is :

$$C_q := [x]^q B_{n+q-1}(x),$$

so that C_q has the value

$$C_q = \sum_{k=0}^{n+q} \binom{n+q}{k} \left(\prod_{j=1}^q (n+k+j) \right) \frac{1 - (-1)^{n+q-k}}{2} B_{n+1+k}$$

On the other hand, the coefficient of x^q in $H_{n+q-1}(x)$ is :

$$\begin{cases} 0 & \text{if } n \geq 1 \\ (-1)^{q+1} q & \text{if } n = 0 \end{cases}$$

As we have :

$$\frac{1 + (-1)^m}{2} B_m = \begin{cases} B_m & \text{if } m \neq 1 \\ 0 & \text{if } m = 1 \end{cases}$$

We get the aimed relationship.

Remark 4.3. I would like to dedicate this modest contribution to the memory of Tom Mike Apostol who passed away on 8 May of this year 2016.(APO2)

□

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